Integrable lattice models for conjugate $A_n^{(1)}$

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Abstract
A new class of $A_n^{(1)}$ integrable lattice models is presented. These are interaction-round-a-face models based on fundamental nimrep graphs associated with the $A_n^{(1)}$ conjugate modular invariants, there being a model for each value of the rank and level. The Boltzmann weights are parametrized by elliptic theta functions and satisfy the Yang–Baxter equation for any fixed value of the elliptic nome q. At $q = 0$, the models provide representations of the Hecke algebra and are expected to lead in the continuum limit to coset conformal field theories related to the $A_n^{(1)}$ conjugate modular invariants.

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1. Introduction

Modular invariants and nimreps constitute important combinatorial objects in the classification of rational conformal field theories. (For an account of these objects, see for example [1, 2].) However, a rational conformal field theory is not completely described by its modular invariant and associated nimrep, and in some cases much additional information can be provided by a corresponding integrable lattice model.

This issue was considered for the case of $A_n^{(1)}$ level-k in [3–5]. As outlined there, for an $A_{n,k}^{(1)}$ modular invariant M and associated nimrep $\mathcal{N}$ there may exist a critical integrable interaction-round-a-face lattice model whose continuum limit provides a realization of the $A_{n,k-1}^{(1)} \times A_{n,1}^{(1)} / A_{n,k}^{(1)}$ coset conformal field theory with modular invariant I $\otimes$ I $\otimes$ M. The main features of such a model are that its spin states take values from the same set as that which labels the rows and columns of the nimrep matrices $\mathcal{N}_\lambda$, that the adjacency condition for spins on neighbouring sites of the lattice is given by the nimrep graph for the fundamental vector representation of $A_{n,k}^{(1)}$, and that the Boltzmann weights at some value of the spectral parameter provide a representation of a certain quotient of the Hecke algebra.

For any modular data, two particularly simple and natural modular invariants are the identity and the conjugate, with nimreps for the former being automatically given by fusion matrices and nimreps for various cases of the latter having recently been given explicitly in

[6, 7]. (Note, however, that if specialised characters are used, for which conjugate representations have the same character, then the identity and conjugate modular invariants both give the same modular invariant partition function.) For the $A_{n,k}^{(1)}$ identity modular invariant, corresponding integrable lattice models of the type outlined in [3–5] are given in [8, 9]. The Boltzmann weights in these models are parametrized by elliptic theta functions of fixed nome q, and it is then specifically the case $q = 0$ which corresponds to criticality and which gives models whose continuum limit leads to coset conformal field theories.

In this paper we present general models for the case of the $A_{n,k}^{(1)}$ conjugate modular invariant. These models are based, therefore, on the fundamental nimrep graphs of [6, 7], and as with the models of [8, 9] their Boltzmann weights are parametrized by elliptic functions of nome q, with $q = 0$ giving models of the type outlined in [3–5]. We describe the nimrep graphs in section 2, present and study the Boltzmann weights for arbitrary q in section 3, and study the Boltzmann weights for $q = 0$ in section 4. In section 5 we briefly discuss some general matters arising from the previous results, including the possibility of relationships between the conjugate $A_{n,k}^{(1)}$ Boltzmann weights and the $B_l^{(1)}$, $C_l^{(1)}$ and $A_l^{(2)}$ Boltzmann weights of [10, 11], the existence of cells which intertwine identity and conjugate $A_{n,k}^{(1)}$ Boltzmann weights and the existence of fused conjugate $A_{n,k}^{(1)}$ models.

2. Graphs for conjugate $A_n^{(1)}$

In this section we explicitly describe the required nimrep matrices and graphs associated with the $A_n^{(1)}$ conjugate modular invariants. This material is entirely based on the results of [6, 7], in which the nimrep matrices were given in terms of their eigenvalues and eigenvectors, and also in terms of certain $A_l^{(1)}$, $B_l^{(1)}$ and $C_l^{(1)}$ fusion matrices.

We are considering the affine Kac–Moody algebra $A_n^{(1)} = \hat{sl}(n + 1)$ of rank n at fixed level k. We shall from now on work with the shifted level, or altitude,

$$g = n + k + 1 \tag{2.1}$$

and denote $A_n^{(1)}$ at shifted level g as $A_{n,g}$. The integrable representations of $A_{n,g}$ are labelled by elements or weights of the Weyl alcove

$$P_{n,g} = \{\lambda \equiv (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \mid \lambda_1 + \cdots + \lambda_n \leqslant g - 1\}. \tag{2.2}$$

In this notation, the identity or vacuum representation is labelled by $(1, \ldots, 1)$ and the n fundamental representations by $(2, 1, \ldots, 1), (1, 2, 1, \ldots, 1), \ldots, (1, \ldots, 1, 2)$. The fundamental representation labelled by $(2, 1, \ldots, 1)$ is referred to as the vector representation and corresponds to a Young diagram comprising a single square. The cardinality of $P_{n,g}$ is $\binom{g-1}{n}$. Conjugation of representations is given by

$$(\lambda_1, \ldots, \lambda_n)^* = (\lambda_n, \ldots, \lambda_1). \tag{2.3}$$

The fusion rule coefficients are denoted by $N_{\lambda\mu}{}^\nu$ ($\lambda, \mu, \nu \in P_{n,g}$), and lead to the fusion matrices $(N_\lambda)_{\mu\nu} = N_{\lambda\mu}{}^\nu$ which form a representation of the fusion ring,

$$N_\lambda N_\mu = \sum_{\nu \in P_{n,g}} N_{\lambda\mu}{}^\nu N_\nu. \tag{2.4}$$

A nimrep is any further collection of nonnegative-integer-entry matrices $\mathcal{N}_\lambda$ ($\lambda \in P_{n,g}$), each with rows and columns labelled by the elements of some set $\mathcal{B}$, which form a representation of the fusion ring,

$$\mathcal{N}_\lambda \mathcal{N}_\mu = \sum_{\nu \in P_{n,g}} N_{\lambda\mu}{}^\nu \mathcal{N}_\nu \tag{2.5}$$

and satisfy $\mathcal{N}_{(1,\ldots,1)} = I$ and $\mathcal{N}_{\lambda^*} = \mathcal{N}_\lambda{}^t$. It is useful to regard nimrep matrices as the adjacency matrices of graphs; that is, the vertices of such a graph are the elements of $\mathcal{B}$ and there are $\mathcal{N}_{\lambda ab}$ edges connecting vertex a to vertex b. In cases of interest in conformal field theory, it is expected that each $A_{n,g}$ modular-invariant matrix M can be associated with at least one nimrep, with the cardinalities of the corresponding $\mathcal{B}$ given by $\mathrm{tr}\, M$.

Of sole interest here are the nimreps related to $A_{n,g}$ conjugation, as studied in detail in [6, 7]. These are associated with the modular-invariant matrix $M_{\mu\nu} = \delta_{\mu,\nu^*}$, and $|\mathcal{B}|$ is therefore the number of self-conjugate representations in $P_{n,g}$. Each such nimrep matrix is symmetric. (The eigenvalues of $\mathcal{N}_\lambda$ are $S_{\lambda\mu}/S_{\lambda,(1\ldots1)}$ for all self-conjugate μ, where S is the $A_{n,g}$ S-matrix, and from the conjugation properties of S these are seen to be real.)

The labelling sets for these $A_{n,g}$ conjugation nimreps are taken as

$$
\mathcal{B} = \begin{cases} \left\{ a \equiv \left(a_1, \ldots, a_{\frac{n+1}{2}}\right) \in \mathbb{N}^{\frac{n+1}{2}} \,\middle|\, a_1 + 2\left(a_2 + \cdots + a_{\frac{n+1}{2}}\right) \leqslant g - 1 \right\} & n \text{ odd} \\ \left\{ a \equiv \left(a_1, \ldots, a_{\frac{n}{2}}\right) \in \mathbb{N}^{\frac{n}{2}} \,\middle|\, 2\left(a_1 + \cdots + a_{\frac{n}{2}-1}\right) + a_{\frac{n}{2}} \leqslant g - 1, \, (-1)^{g + a_{\frac{n}{2}}} = 1 \right\} & n \text{ even.} \end{cases}
\tag{2.6}
$$

For $n = 1$ and 2, (2.6) should be interpreted as

$$
\mathcal{B} = \begin{cases} \{1, 2, \ldots, g - 1\} & n = 1 \\ \{1, 3, \ldots, g - 2\} & n = 2 \quad g \text{ odd} \\ \{2, 4, \ldots, g - 2\} & n = 2 \quad g \text{ even.} \end{cases}
\tag{2.7}
$$

The labelling sets can also be generated using $P_{n,g}$ as

$$
\mathcal{B} = \begin{cases} \left\{ \left(2a_1, a_2 \ldots, a_{\frac{n+1}{2}}\right) \,\middle|\, \left(a_1, \ldots, a_{\frac{n+1}{2}}\right) \in P_{\frac{n+1}{2}, \lfloor\frac{g+1}{2}\rfloor} \right\} \bigcup & \\ \quad \left\{ \left(2a_1 - 1, a_2 \ldots, a_{\frac{n+1}{2}}\right) \,\middle|\, \left(a_1, \ldots, a_{\frac{n+1}{2}}\right) \in P_{\frac{n+1}{2}, \lfloor\frac{g}{2}\rfloor + 1} \right\} & n \text{ odd} \\ \left\{ \left(a_1, \ldots, a_{\frac{n}{2}-1}, 2a_{\frac{n}{2}} - 1\right) \,\middle|\, \left(a_1, \ldots, a_{\frac{n}{2}}\right) \in P_{\frac{n}{2}, \frac{g+1}{2}} \right\} & n \text{ even} \quad g \text{ odd} \\ \left\{ \left(a_1, \ldots, a_{\frac{n}{2}-1}, 2a_{\frac{n}{2}}\right) \,\middle|\, \left(a_1, \ldots, a_{\frac{n}{2}}\right) \in P_{\frac{n}{2}, \frac{g}{2}} \right\} & n \text{ even} \quad g \text{ even} \end{cases}
\tag{2.8}
$$

from which it follows that $|\mathcal{B}| = \binom{\lfloor\frac{g-1}{2}\rfloor}{\frac{n+1}{2}} + \binom{\lfloor\frac{g}{2}\rfloor}{\frac{n+1}{2}}$ for n odd, and $|\mathcal{B}| = \binom{\lfloor\frac{g-1}{2}\rfloor}{\frac{n}{2}}$ for n even. We note that the labelling used here differs from that of [7] for the case n even with g odd, but is otherwise essentially equivalent to that of [6] and [7].

Of primary interest here are the nimrep matrices for the fundamental vector representation $(2, 1, \ldots, 1)$, whose entries are given by

$$
\mathcal{N}_{(2,1,\ldots,1)ab} = \begin{cases} \displaystyle\sum_{i=1}^{\frac{n+1}{2}} (\delta_{a-b,\varepsilon_i} + \delta_{b-a,\varepsilon_i}) & n \text{ odd} \\ \displaystyle\sum_{i=1}^{\frac{n}{2}} (\delta_{a-b,\varepsilon_i} + \delta_{b-a,\varepsilon_i}) + \left(1 - \delta_{a_{\frac{n}{2}},1}\right) \delta_{a,b} & n \text{ even} \end{cases}
\tag{2.9}
$$

where $\varepsilon_i \in \mathbb{Z}^{\lfloor\frac{n+1}{2}\rfloor}$ are defined as:

n odd	n even
$\varepsilon_1 = (1, 0, \ldots, 0)$	$\varepsilon_1 = (1, 0, \ldots, 0)$
$\varepsilon_2 = (-1, 1, 0, \ldots, 0)$	$\varepsilon_2 = (-1, 1, 0, \ldots, 0)$
$\varepsilon_3 = (0, -1, 1, 0, \ldots, 0)$	$\varepsilon_3 = (0, -1, 1, 0, \ldots, 0)$
$\vdots$	$\vdots$
$\varepsilon_{\frac{n-1}{2}} = (0, \ldots, 0, -1, 1, 0)$	$\varepsilon_{\frac{n}{2}-1} = (0, \ldots, 0, -1, 1, 0)$
$\varepsilon_{\frac{n+1}{2}} = (0, \ldots, 0, -1, 1)$	$\varepsilon_{\frac{n}{2}} = (0, \ldots, 0, -1, 2)$

(2.10)

For $n = 1$ we take $\varepsilon_1 = 1$ and for $n = 2$ we take $\varepsilon_1 = 2$. We shall also use $\varepsilon_i \in \mathbb{Z}^{\lfloor \frac{n+1}{2} \rfloor}$ with $i \leqslant 0$, defined by

$$\varepsilon_0 = (0, \ldots, 0) \quad (n \text{ even}), \qquad \varepsilon_i = -\varepsilon_{-i} \quad i < 0. \tag{2.11}$$

We shall denote the graph corresponding to $\mathcal{N}_{(2,1,\ldots,1)}$ as $\mathcal{G}_{n,g}$, or simply $\mathcal{G}$, and its adjacency matrix as G. Thus the set of vertices of $\mathcal{G}$ is $\mathcal{B}$, $G = \mathcal{N}_{(2,1,\ldots,1)}$ and any edge of $\mathcal{G}$ lies along ε_i, for some i. We see from (2.9) that G is symmetric with each entry either 0 or 1, so that $\mathcal{G}$ is an unoriented graph with only single edges. We shall denote the set of r-step paths on $\mathcal{G}$ by $\mathcal{G}^r$,

$$\mathcal{G}^r = \left\{ (a_0, \ldots, a_r) \in \mathcal{B}^{r+1} \,\middle|\, \prod_{j=0}^{r-1} G_{a_j a_{j+1}} = 1 \right\}. \tag{2.12}$$

Some examples of $\mathcal{G}$ for low rank ($n \leqslant 4$) or low level ($k \leqslant 3$) are given in the following figures, in which brackets and commas are omitted from the vertex labels for simplicity, and in which $\mathcal{A}_l$ and $\mathcal{D}_l$ are the A and D Dynkin diagrams and $\mathcal{T}_l$ is the 'tadpole' graph (in which an end vertex of $\mathcal{A}_l$ becomes self-adjacent):

$$\mathcal{G}_{n,n+3} = \begin{cases} & \approx \mathcal{D}_{\frac{n+5}{2}} \qquad n \text{ odd} \\ & \\ & \approx \mathcal{T}_{\frac{n}{2}+1} \qquad n \text{ even} \end{cases}$$

$$\mathcal{G}_{n,n+4} = \begin{cases} & n \text{ odd} \\ & \\ & \approx \mathcal{G}_{2,n+4} \qquad n \text{ even.} \end{cases}$$

We note that $\mathcal{B}$ can be regarded as being a subset of the Weyl alcove $P^C_{\frac{n+1}{2},g}$ for n odd or $P^B_{\frac{n}{2},g}$ for n even, and that with this identification we have

$$\mathcal{N}_{(2,1,\ldots,1)ab} = \begin{cases} N^{C}_{(2,1,\ldots,1)a}{}^{b} & n \text{ odd} \\ N^{B}_{(2,1,\ldots,1)a}{}^{b} & n \text{ even} \end{cases} \tag{2.13}$$

where B and C refer to the algebras $C^{(1)}_{\frac{n+1}{2}}$ and $B^{(1)}_{\frac{n}{2}}$, each at shifted level g (or levels $g - \frac{n+1}{2} - 1$ and $g - n + 1$ respectively), and $N^{C}_{ab}{}^{c}$ and $N^{B}_{ab}{}^{c}$ are their fusion coefficients (with $(2, 1, \ldots, 1)$ on the LHS of (2.13) a weight of the $A_{n,g}$ Weyl alcove $P_{n,g}$, and $(2, 1, \ldots, 1)$ on the RHS a weight of the Weyl alcoves $P^C_{\frac{n+1}{2},g}$ or $P^B_{\frac{n}{2},g}$).

For the case n odd and the case n even with g even, there exist involutions J of $\mathcal{B}$,

$$J\left(a_1, \cdots, a_{\lfloor \frac{n+1}{2} \rfloor}\right) = \begin{cases} \left(g - a_1 - 2a_2 - \cdots - 2a_{\frac{n+1}{2}}, a_2, \ldots, a_{\frac{n+1}{2}}\right) & n \text{ odd} \\ \left(a_{\frac{n}{2}-1}, \ldots, a_1, g - 2a_1 - \cdots - 2a_{\frac{n}{2}-1} - a_{\frac{n}{2}}\right) & n \text{ even} \quad g \text{ even} \end{cases} \tag{2.14}$$

for which

$$\mathcal{N}_{\lambda, Ja, Jb} = \mathcal{N}_{\lambda ab} \tag{2.15}$$

for any $\lambda \in P_{n,g}$ and $a, b \in \mathcal{B}$.

In section 3 it will be convenient to use orthogonal coordinates for each $a \in \mathcal{B}$, defined as

$$\bar{a}_i = \begin{cases} \displaystyle\sum_{j=i}^{\frac{n+1}{2}} a_j & n \text{ odd} \\ \displaystyle\sum_{j=i}^{\frac{n}{2}-1} a_j + \frac{a_{\frac{n}{2}}}{2} & n \text{ even} \end{cases} \qquad i > 0, \qquad \bar{a}_i = -\bar{a}_{-i} \quad i < 0, \qquad \bar{a}_0 = -\frac{1}{2}. \tag{2.16}$$

We see from (2.9) that if $\left(a_1, \ldots, a_{\lfloor \frac{n+1}{2} \rfloor}\right)$ is relabelled as $\left(\bar{a}_1, \ldots, \bar{a}_{\lfloor \frac{n+1}{2} \rfloor}\right)$, then the nonzero edges of $\mathcal{G}$ lie along $\pm\bar{\varepsilon}_1, \ldots, \pm\bar{\varepsilon}_{\lfloor \frac{n+1}{2} \rfloor}$, where $(\bar{\varepsilon}_i)_j = \delta_{ij}$.

3. Boltzmann weights for conjugate $A_n^{(1)}$

In this section we present Boltzmann weights for integrable lattice models based on the conjugate $A_{n,g}$ graphs $\mathcal{G}$. In these models, a spin is attached to each site of a two-dimensional square lattice, with the possible states of each spin being the vertices of $\mathcal{G}$ and there being a lattice adjacency condition stipulating that, in any assignment of spin states to the lattice, the states on each pair of nearest-neighbour sites must correspond to an edge of $\mathcal{G}$. These models are interaction-round-a-face models, so that Boltzmann weights $W\binom{a\ \ d}{b\ \ c}\big|u)$ are associated with sets of four spin states, a, b, c, d, adjacent around a square face, where $u \in \mathbb{C}$ is the spectral parameter. The partition function of the model is then the sum, over all possible spin assignments, of products of Boltzmann weights over all square faces of the lattice. The Boltzmann weights introduced here satisfy various properties, and in particular the Yang–Baxter equation which ensures the integrability of the model. The notation we use is similar to that of [8–11].

We define

$$[u]_1 = \vartheta_1\left(\frac{u\pi}{g}, q\right) \qquad [u]_4 = \vartheta_4\left(\frac{u\pi}{g}, q\right) \tag{3.1}$$

where ϑ_1 and ϑ_4 are, up to a factor of $2q^{1/4}$ in ϑ_1, elliptic theta functions of fixed nome q, with $|q| < 1$,

$$\vartheta_1(u, q) = \sin u \prod_{n=1}^{\infty}(1 - 2q^{2n}\cos 2u + q^{4n})(1 - q^{2n})$$

$$\vartheta_4(u, q) = \prod_{n=1}^{\infty}(1 - 2q^{2n-1}\cos 2u + q^{4n-2})(1 - q^{2n}). \tag{3.2}$$

The Boltzmann weights are then given as

$$W\begin{pmatrix} a & a + \varepsilon_i \\ a + \varepsilon_i & a + 2\varepsilon_i \end{pmatrix} u = \frac{[\lambda - u]_4[1 - u]_1}{[\lambda]_4[1]_1} \qquad i \neq 0$$

$$W\begin{pmatrix} a & a + \varepsilon_i \\ a + \varepsilon_i & a + \varepsilon_i + \varepsilon_j \end{pmatrix} u = \frac{[\lambda - u]_4[\bar{a}_i - \bar{a}_j + u]_1}{[\lambda]_4[\bar{a}_i - \bar{a}_j]_1} \qquad i \neq \pm j$$

$$W\begin{pmatrix} a & a + \varepsilon_j \\ a + \varepsilon_i & a + \varepsilon_i + \varepsilon_j \end{pmatrix} u = \left(\frac{[\bar{a}_i - \bar{a}_j - 1]_1[\bar{a}_i - \bar{a}_j + 1]_1}{[\bar{a}_i - \bar{a}_j]_1^2}\right)^{1/2} \frac{[\lambda - u]_4[u]_1}{[\lambda]_4[1]_1} \qquad i \neq \pm j$$

$$W\begin{pmatrix} a & a + \varepsilon_j \\ a + \varepsilon_i & a \end{pmatrix} u = \sigma_{ij}(-1)^{n+1}(\psi_{a,i}\psi_{a,j})^{1/2}\frac{[\bar{a}_i + \bar{a}_j + 1 - \lambda + u]_4[u]_1}{[\lambda]_4[\bar{a}_i + \bar{a}_j + 1]_1} \qquad i \neq j$$

$$W\begin{pmatrix} a & a + \varepsilon_i \\ a + \varepsilon_i & a \end{pmatrix} u = \frac{[\lambda - u]_4[2\bar{a}_i + 1 + u]_1 + (-1)^{n+1}\psi_{a,i}[2\bar{a}_i + 1 - \lambda + u]_4[u]_1}{[\lambda]_4[2\bar{a}_i + 1]_1}$$

$$= \frac{[\lambda + u]_4[2\bar{a}_i + 1 - 2\lambda + u]_1 + (-1)^n\phi_{a,i}[2\bar{a}_i + 1 - \lambda + u]_4[u]_1}{[\lambda]_4[2\bar{a}_i + 1 - 2\lambda]_1} \tag{3.3}$$

where

$$\lambda = \frac{n + 1}{2} \qquad \sigma_{ij} = \begin{cases} \text{sign}(i)\,\text{sign}(j) & n \text{ odd} \\ 1 & n \text{ even} \end{cases} \tag{3.4}$$

$$\psi_{a,i} = \frac{\psi_{a+\varepsilon_i}}{\psi_a} \tag{3.5}$$

$$\psi_a = \begin{cases} \displaystyle\prod_{i=1}^{\frac{n+1}{2}} [\bar{a}_i]_4[\bar{a}_i]_1 \prod_{1\leqslant i<j\leqslant \frac{n+1}{2}} [\bar{a}_i - \bar{a}_j]_1[\bar{a}_i + \bar{a}_j]_1 & n \text{ odd} \\[2em] \displaystyle\prod_{i=1}^{\frac{n}{2}} \frac{[2\bar{a}_i]_1}{[\bar{a}_i]_4} \prod_{1\leqslant i<j\leqslant \frac{n}{2}} [\bar{a}_i - \bar{a}_j]_1[\bar{a}_i + \bar{a}_j]_1 & n \text{ even} \end{cases} \tag{3.6}$$
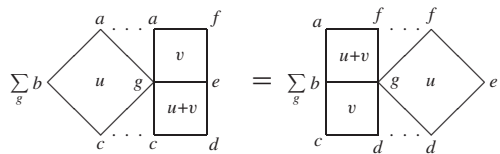
and

$$\phi_{a,i} = \sum_{\substack{b=a+\varepsilon_j \in \mathcal{B} \\ (G_{ab}=1, j\neq i)}} \frac{[\bar{a}_i + \bar{a}_j + 1 - 2\lambda]_1}{[\bar{a}_i + \bar{a}_j + 1]_1} \, \psi_{a,j}. \tag{3.7}$$

Substituting (3.6) into (3.5), $\psi_{a,i}$ for $i \neq 0$ can be written explicitly as
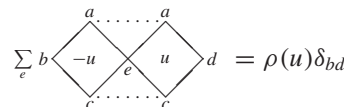
$$\psi_{a,i} = \begin{cases} \dfrac{[\bar{a}_i + 1]_4[\bar{a}_i + 1]_1}{[\bar{a}_i]_4[\bar{a}_i]_1} \displaystyle\prod_{j\in\{\pm 1,\dots,\pm\frac{n+1}{2}\}/\{\pm i\}} \dfrac{[\bar{a}_i + \bar{a}_j + 1]_1}{[\bar{a}_i + \bar{a}_j]_1} & n \text{ odd} \\[2em] \dfrac{[\bar{a}_i]_4[2\bar{a}_i + 2]_1}{[\bar{a}_i + 1]_4[2\bar{a}_i]_1} \displaystyle\prod_{j\in\{\pm 1,\dots,\pm\frac{n}{2}\}/\{\pm i\}} \dfrac{[\bar{a}_i + \bar{a}_j + 1]_1}{[\bar{a}_i + \bar{a}_j]_1} & n \text{ even}. \end{cases} \tag{3.8}$$

For each set $(a, b, c, d, a) \in \mathcal{G}^4$ of spin states adjacent around a square face, there exists a unique assignment of i or of i and j in exactly one of the five equations of (3.3) which defines the Boltzmann weight $W\left(\begin{smallmatrix} a & d \\ b & c \end{smallmatrix}\middle| u\right)$. The two forms of $W\left(\begin{smallmatrix} a & a+\varepsilon_i \\ a+\varepsilon_i & a \end{smallmatrix}\middle| u\right)$ in (3.3) are equivalent, except if $[2\bar{a}_i + 1]_1 = 0$ (which occurs for example for n even with $i = 0$) in which case only the second form can be used, or if $[2\bar{a}_i + 1 - 2\lambda]_1 = 0$ in which case only the first form can be used. The equivalence of these two forms can be proved using the same method as that used in [10] to prove a similar equivalence for the Boltzmann weights of that paper.

The Boltzmann weights of (3.3) satisfy the Yang–Baxter equation

$$\sum_{\substack{g\in\mathcal{B} \\ (G_{ag}G_{gc}G_{ge}=1)}} W\begin{pmatrix} a & g \\ b & c \end{pmatrix}u\,\,W\begin{pmatrix} g & e \\ c & d \end{pmatrix}u+v\,\,W\begin{pmatrix} a & f \\ g & e \end{pmatrix}v$$

$$= \sum_{\substack{g\in\mathcal{B} \\ (G_{bg}G_{fg}G_{gd}=1)}} W\begin{pmatrix} b & g \\ c & d \end{pmatrix}v\,\,W\begin{pmatrix} a & f \\ b & g \end{pmatrix}u+v\,\,W\begin{pmatrix} f & e \\ g & d \end{pmatrix}u$$

$$\tag{3.9}$$

for all $(a, b, c, d, e, f, a) \in \mathcal{G}^6$ and $u, v \in \mathbb{C}$, and the inversion relation

$$\sum_{\substack{e\in\mathcal{B} \\ (G_{ae}G_{ec}=1)}} W\begin{pmatrix} a & e \\ b & c \end{pmatrix}-u\,\,W\begin{pmatrix} a & d \\ e & c \end{pmatrix}u = \rho(u)\delta_{bd}$$

$$\tag{3.10}$$

for all $(a, b, c, d, a) \in \mathcal{G}^4$ and $u \in \mathbb{C}$, where

$$\rho(u) = \frac{[\lambda - u]_4 [\lambda + u]_4 [1 - u]_1 [1 + u]_1}{[\lambda]_4{}^2 [1]_1{}^2}. \tag{3.11}$$

That the Yang–Baxter equation and inversion relation are satisfied by the conjugate $A_{n,g}$ weights of (3.3) can be proved using an approach similar to that used in [10] to prove that these equations are satisfied by the B and C weights of that paper. Although this constitutes a long proof, it is the only one currently known to us.

It is straightforward to check that the weights of (3.3) also satisfy the initial condition

$$W \begin{pmatrix} a & d \\ b & c \end{pmatrix} 0 \end{pmatrix} = \delta_{bd} \tag{3.12}$$

reflection symmetry,

$$W \begin{pmatrix} a & d \\ b & c \end{pmatrix} u \end{pmatrix} = W \begin{pmatrix} a & b \\ d & c \end{pmatrix} u \end{pmatrix} = W \begin{pmatrix} c & d \\ b & a \end{pmatrix} u \end{pmatrix} \tag{3.13}$$

and graph symmetry for n odd,

$$W \begin{pmatrix} a & d \\ b & c \end{pmatrix} u \end{pmatrix} = W \begin{pmatrix} Ja & Jd \\ Jb & Jc \end{pmatrix} u \end{pmatrix} \qquad (n \text{ odd}) \tag{3.14}$$

with J as given in (2.14). The weights are not invariant under J for n even, g even.

In section 4, it will be useful to consider the Boltzmann weights as face transfer matrices. These are matrices $X_1(u), \ldots, X_r(u)$, for some fixed $r \in \mathbb{N}$, whose rows and columns are labelled by elements of $\mathcal{G}^{r+1}$ and whose entries are defined by

$$X_j(u)_{(a_0,\ldots,a_{r+1}),(b_0,\ldots,b_{r+1})} = W \begin{pmatrix} a_{j-1} & b_j \\ a_j & a_{j+1} \end{pmatrix} u \end{pmatrix} \prod_{\substack{k=0 \\ k \neq j}}^{r+1} \delta_{a_k b_k}. \tag{3.15}$$

It follows from the Yang–Baxter equation (3.9) that these matrices satisfy

$$X_j(u) X_{j+1}(u + v) X_j(v) = X_{j+1}(v) X_j(u + v) X_{j+1}(u) \tag{3.16}$$

and from the inversion relation (3.10) that they satisfy

$$X_j(-u) X_j(u) = \rho(u) I. \tag{3.17}$$

It is also seen immediately from (3.15) that

$$X_j(u) X_k(v) = X_k(v) X_j(u) \qquad |j - k| \geqslant 2. \tag{3.18}$$

4. The critical case

In this section we consider the Boltzmann weights of section 3 at $q = 0$, where q is the elliptic nome of (3.1), this being expected to correspond to criticality. In particular, we examine the various general properties outlined in [3–5].

For $q = 0$ we now have

$$[u]_1 = \sin\left(\frac{u\pi}{g}\right) \equiv s(u) \qquad [u]_4 = 1 \tag{4.1}$$

throughout (3.3)–(3.8). It can be shown, incidentally, that ψ_a are now entries of the Perron–Frobenius eigenvector of G,

$$\sum_{b \in \mathcal{B}} G_{ab} \psi_b = \frac{s(n + 1)}{s(1)} \psi_a. \tag{4.2}$$

Defining

$$U_j = X_j(1) \tag{4.3}$$

it follows using standard trigonometric identities on $W\left(\begin{smallmatrix} a & a+\varepsilon_i \\ a+\varepsilon_i & a+\varepsilon_i+\varepsilon_j \end{smallmatrix} \middle| u\right)$ and $W\left(\begin{smallmatrix} a & a+\varepsilon_i \\ a+\varepsilon_i & a \end{smallmatrix} \middle| u\right)$ in (3.3) that the critical face transfer matrices can be decomposed as

$$X_j(u) = \frac{s(1-u)}{s(1)} I + \frac{s(u)}{s(1)} U_j. \tag{4.4}$$

Substituting this into (3.16)–(3.18), it then follows that U_j form a matrix representation of the Hecke algebra,

$$
\begin{aligned}
U_j^2 &= 2\cos(\pi/g)U_j \\
U_j U_{j+1} U_j - U_j &= U_{j+1} U_j U_{j+1} - U_{j+1} \\
U_j U_k &= U_k U_j \qquad |j-k| \geqslant 2.
\end{aligned}
\tag{4.5}
$$

The critical Boltzmann weights are also expected to satisfy the trace property,

$$\frac{1}{2\cos(\pi/g)} \sum_{\substack{c \in \mathcal{B} \\ (G_{ac}G_{cb}=1)}} W\left(\begin{matrix} a & c \\ c & b \end{matrix} \middle| 1\right) = \mathcal{N}_{(1,2,1,\dots,1)ab} \tag{4.6}$$

for all $a, b \in \mathcal{B}$ with $(G^2)_{ab} \neq 0$, the Markov property,

$$\sum_{\substack{c \in \mathcal{B} \\ (G_{ac}=1)}} W\left(\begin{matrix} b & a \\ a & c \end{matrix} \middle| u\right) \psi_c = \frac{s(n+1-u)}{s(1)} \psi_a \tag{4.7}$$

for all $(a, b) \in \mathcal{G}^1$ and $u \in \mathbb{C}$, and the quantum group antisymmetrizer property,

$$\sum_{\sigma \in S_{n+2}} \prod_{\tau_j \in T_\sigma} Y_j = 0 \tag{4.8}$$

where $Y_j = e^{\frac{i\pi}{g}} U_j - I$, S_n is the symmetric group on n objects, $1, 2, \dots, n$, and T_σ is any minimal-length set of transpositions $\tau_j \equiv (j \leftrightarrow j+1)$ whose product in a particular order is the permutation σ, with the product in (4.8) taken in the same order. (If σ is the identity permutation, then $T_\sigma = \emptyset$ and $\prod_{\tau_j \in T_\sigma} Y_j$ is taken as the identity matrix. Also, it follows from (4.5) that Y_j satisfy the Braid relations $Y_j Y_{j+1} Y_j = Y_{j+1} Y_j Y_{j+1}$ and $Y_j Y_k = Y_k Y_j$ for $|j-k| \geqslant 2$, which implies that $\prod_{\tau_j \in T_\sigma} Y_j$ is independent of the choice of T_σ for given σ.) The imposition of the quantum group antisymmetrizer property (4.8) on the Hecke algebra (4.5) can be regarded as taking a quotient of that algebra.

We have checked that properties (4.6)–(4.8) are satisfied by the critical conjugate $A_{n,g}$ Boltzmann weights for a variety of choices of n and g, and we believe that these properties hold for all n and g. The significance of these properties is discussed in [3–5].

5. Discussion

We have presented Boltzmann weights for integrable lattice models based on fundamental conjugate $A_{n,g}$ nimrep graphs. We derived these weights through direct and detailed consideration of the Yang–Baxter equation as applied to these graphs, but it would clearly be of interest to obtain an alternative systematic algebraic procedure for generating the weights which reflects one or more of the procedures used in [6, 7] to generate the nimreps. Such a procedure would be expected to reveal relationships between the conjugate $A_{n,g}$ Boltzmann

weights and the $B_l^{(1)}$, $C_l^{(1)}$ and $A_l^{(2)}$ Boltzmann weights of [10, 11], reflecting such relationships at the level of the nimreps.

It is also expected, at least at criticality, that there exist cells which intertwine the conjugate $A_{n,g}$ Boltzmann weights with the identity $A_{n,g}$ Boltzmann weights of [8, 9]. If such cells were obtained and shown to satisfy intertwining and inversion equations (specifically (4.6a,b,c) of [3]), it would then follow immediately that key relations known to be satisfied by the identity $A_{n,g}$ Boltzmann weights, including the Yang–Baxter equation and the quantum group antisymmetrizer property, are also satisfied by the conjugate $A_{n,g}$ weights.

The lattice models given here are related to the fundamental vector representation $(2, 1, \ldots, 1)$ of $A_{n,g}$, with horizontal and vertical adjacency conditions on the lattice both specified by $\mathcal{N}_{(2,1,\ldots,1)}$. In the case of the identity $A_{n,g}$ lattice models of [8, 9], an $A_{n,g}$ fusion procedure was applied in [12–14] to the fundamental vector Boltzmann weights to construct weights which still satisfy the Yang–Baxter equation and have horizontal and vertical adjacency conditions given by N_λ and N_μ for any $\lambda, \mu \in P_{n,g}$. (In fact the graphs can then contain multiple edges between two given vertices, which requires the introduction of additional spins on the edges of the lattice.) And it is expected, at least at criticality, that essentially the same fusion procedure can be applied to the conjugate $A_{n,g}$ fundamental vector weights to construct conjugate $A_{n,g}$ weights with horizontal and vertical adjacency conditions given by arbitrary $\mathcal{N}_\lambda$ and $\mathcal{N}_\mu$. An important component of this procedure is the construction of projection matrices $Q_\lambda(a, b)$, for each $\lambda \in P_{n,g}$ and $a, b \in \mathcal{B}$. The rows and columns of $Q_\lambda(a, b)$ are labelled by the r-step paths on $\mathcal{G}_{n,g}$ between a and b, where r is the number of boxes in the Young diagram corresponding to λ, $r = \sum_{i=1}^n i(\lambda_i - 1)$, and the rank of the projector $Q_\lambda(a, b)$ is $\mathcal{N}_{\lambda ab}$. (For the case $\lambda = (1, 2, 1, \ldots, 1)$, this projector is given by $Q_\lambda(a, b)_{cd} = W\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \middle| 1\right) / (2\cos(\pi/g))$ and the trace property (4.6) is the equation for its rank, since the rank of a projector is simply its trace.) If it can be shown that the $A_{n,g}$ fusion procedure of [12–14] is indeed applicable to the critical conjugate $A_{n,g}$ Boltzmann weights, then an immediate corollary would be the nonnegativity of each $\mathcal{N}_{\lambda ab}$ for conjugate $A_{n,g}$ (since these integers can then be interpreted as ranks of projectors), this nonnegativity not having been proved in general in [6, 7].

We also note that an approach to the generation of modular invariants and nimreps, in which the nimrep matrices are guaranteed to have nonnegative entries, is that of α-induction, as described in [15, 16]. If this were applied to the case of the $A_{n,g}$ conjugate modular invariant, the critical conjugate $A_{n,g}$ Boltzmann weights would be used together with intertwiner cells to construct braided subfactors which, through α-induction, would be expected to lead to the nimreps of [6, 7].

For a few special cases of n and g, other integrable interaction-round-a-face lattice models are known which are also based on the fundamental vector conjugate $A_{n,g}$ graphs. For example, it can be seen in the figures in section 2 that $\mathcal{G}_{n,n+3}$, n odd, corresponds to a D Dynkin diagram, with other models based on these having been given in [17], and that $\mathcal{G}_{2,g}$, g even, and $\mathcal{G}_{n,n+4}$, n even, both correspond to 'dilute' A Dynkin diagrams, for which other models were given in [18], but these other models differ intrinsically from the conjugate $A_{n,g}$ models. In particular, the critical Boltzmann weights of [17] lead to relations of the type (4.6)–(4.8), but the data in these relations then pertain to $A_{1,n+3}D_{\frac{n+5}{2}}$ nimreps as distinct from $A_{n,n+3}$ conjugate nimreps, while the critical Boltzmann weights of [18] lead to representations of the dilute Temperley–Lieb algebra as opposed to the Hecke algebra. (A further, but trivial, example is $\mathcal{G}_{1,g}$, which corresponds to the A_{g-1} Dynkin diagram. Models based on these graphs were given in [19] and as the $n = 1$ case of the identity $A_{n,g}$ models of [8, 9], and these are equivalent to the conjugate $A_{1,g}$ models, but this corresponds to the fact that $A_{1,g}$ conjugation is simply the identity, so that all objects related to the conjugate are equivalent to those already known for the identity.) When deriving the Boltzmann weights for conjugate $A_{n,g}$, we observed in

various cases that by imposing the properties of section 4 at criticality, the solution to the Yang–Baxter equation was unique, up to unimportant normalization, gauge transformation, shifts in the spectral parameter or relabelling of the graph. We thus conjecture that for any n and g, the Boltzmann weights given here constitute an essentially unique solution of the Yang–Baxter equation based on $\mathcal{G}_{n,g}$ which at criticality satisfies the properties of section 4 for conjugate $A_{n,g}$.

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